On Hardy fields and models of $\mathbb{R}_{\text{exp}}$

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1 Setting

I write $\mathcal{L}_{\text{exp}} = \langle 0, 1, +, -, \cdot, \mathrm{e}^{\cdot}, < \rangle$ for the language of ordered exponential rings, $\mathbb{R}_{\text{exp}}$ for the the real ordered exponential field, and T_{exp} for its elementary theory. Recall that T_{exp} is model complete in $\mathcal{L}_{\text{exp}}$ by Wilkie's theorem, and that $\mathbb{R}_{\text{exp}}$ is o-minimal.

I start with a few preliminary results that I actually didn't know were true in general.

Lemma 1. *Let κ be an infinite cardinal and let $\mathcal{L}$ be a first-order language. An $\mathcal{L}$-structure $\mathcal{M} = (M, \ldots)$ is κ-saturated if and only if for all subsets $A \subseteq M$ with $|A| < \kappa$, all 1-types $p(v_1) \in S_A^1(\mathcal{L})$ over A are satisfiable in $\mathcal{M}$.*

Proposition 1. *Let κ be an infinite cardinal. Let $\mathcal{M} = (M, <, \ldots)$ be an o-minimal structure with such that $(M, <)$ is a dense linear order without endpoints. Assume that for all subsets $L, R \subseteq M$ with cardinality $|L|, |R| < \kappa$ and with $L < R$, there is an $m \in M$ with $L < m < R$. Then $\mathcal{M} = (M, <, \ldots)$ is κ-saturated.*

Corollary 1. *Let κ be an infinite uncountable cardinal. The ordered exponential field $\mathbf{No}(\kappa)$ of surreal numbers of length / birth day $< \kappa$ with Gonshor's exponential function is κ-saturated.*

Proof. The underlying ordered set is κ-saturated by definition of surreal numbers and by a simple fact (found in [1, Chapter 1]): if L and R are sets of surreal numbers with $L < R$, then the simplest number $\{L|R\}$ with

$$L < \{L|R\} < R$$

has birthday $\mathrm{bd}(\{L|R\}) \leqslant \sup \{\mathrm{bd}(a) + 1 : a \in L \cup R\}$.

We also know, by a result of Ehrlich and van den Dries [2], that $\exp(\mathbf{No}(\kappa)) = \mathbf{No}(\kappa)^{>0}$ and that $(\mathbf{No}(\kappa), \exp)$ can be expanded into a model of $\mathbb{R}_{\text{an,exp}}$. Thus $(\mathbf{No}(\kappa), +, \times, \exp)$ is o-minimal. □

Corollary 2. *Each model of T_{exp} elementarily embeds into a $(\mathbf{No}(\kappa), \exp)$ for large enough κ.*

I was asking myself the following questions:

Question 1. Let $\mathcal{H}$ be a Hardy field containing $\mathbb{R}$. Assume that $\mathcal{H}$ is real-closed and closed under exp and log. Is $(\mathcal{H}, +, \cdot, \exp)$ an elementary expansion of $\mathbb{R}_{\text{exp}}$?

Question 2. Let $\mathcal{H}$ be a Hardy field containing $\mathbb{R}$ and closed under exp. Does $(\mathcal{H}, +, \cdot, \exp)$ embed into an elementary extension of $\mathbb{R}_{\text{exp}}$?

Lou found answers to those questions, which I next explain.

2 First question

In order to answer the first question, in the negative, we require three objects:

We write L for Hardy's field of *logarithmico-exponential functions*, i.e. germs at $+\infty$ that can be obtained as compositions of exp, log, and semialgebraic functions $(a, +\infty) \longrightarrow \mathbb{R}$. In other words, this is the closure of the field $\mathbb{R}(\mathrm{id})$ of rational functions under real closure, exp and log, which Hardy showed to be a Hardy field.

Let $\mathcal{H}_{\exp}$ denote the Hardy field of germs at $+\infty$ of $\mathbb{R}_{\exp}$-definable functions $\mathbb{R} \longrightarrow \mathbb{R}$, allowing parameters. Note that given a positive infinite germ $f \in \mathcal{H}_{\exp}^{>\mathbb{R}}$, its functional inverse f^{inv} is also definable with parameters in $\mathbb{R}_{\exp}$, so $f^{\mathrm{inv}} \in \mathcal{H}_{\exp}$.

Let $\mathcal{R} := (R, +, \cdot, \exp)$ be non-standard, i.e. a proper elementary extension of $\mathbb{R}_{\exp}$. Fix an $\xi \in R$ with $\xi > \mathbb{R}$. This exists since $\mathbb{R}$ is the largest archimedean ordered field.

Proposition 2. *There is a unique elementary embedding* $\mathrm{ev}_\xi : \mathcal{H}_{\exp} \longrightarrow \mathcal{R}$ *which commutes with* $\mathcal{L}_{\exp}$*-definable functions* $\mathbb{R}^k \longrightarrow \mathbb{R}$ / $R^k \longrightarrow R$ *with parameters in* $\mathbb{R}$ *and sends the germ* id *of the identity function onto* ξ.

Proof. Cheking all details is a bit tedious but I think this is a well-known result. I just give the definition. Fix a representative f of a germ in $\mathcal{H}_{\exp}$. There is a defining formula $\varphi_0[\bar{a}, v_1, v_2]$ for f with parameters $\bar{a} \in \mathbb{R}^m$ and a real number c_0 such that for all $t > c_0$, the number $f(t)$ is unique with $\mathbb{R}_{\exp} \vDash \varphi_0[\bar{a}, t, f(t)]$. Let $\varphi_1[\bar{b}, v_1, v_2]$ be a second defining formula with parameters $\bar{b} \in \mathbb{R}^n$ satisfying the same relation, with respect to the same germ, for a possibly distinct $c_1 \in \mathbb{R}$. In particular, we have

$$\mathbb{R}_{\exp} \vDash \forall t, (t > \max(c_0, c_1) \longrightarrow (\forall y, (\varphi_0[\bar{a}, t, y] \longleftrightarrow \varphi_1[\bar{b}, t, y]))). \quad (1)$$

Recall that $\mathbb{R}_{\exp}$ is o-minimal, so $\mathbb{R}_{\exp} \preccurlyeq \mathcal{H}_{\exp}$ is an elementary embedding. We have $\xi > c_0$, c_1, so by (1), the unique element y_ξ of $\mathcal{R}$ with $\mathcal{R} \vDash \varphi_0[\bar{a}, \xi, y_\xi]$ is also the unique element of $\mathcal{R}$ satisfying $\varphi_1[\bar{b}, \xi, y_\xi]$. We define $\mathrm{ev}_\xi(f)$ to be that element y_ξ.

Similar arguments show that ev_ξ commutes with $\mathbb{R}_{\exp}$-definable functions $\mathbb{R}^k \longrightarrow \mathbb{R}$, whence in particular that it is an embedding in $\mathcal{L}_{\exp}$, hence an elementary embedding by model completeness. □

Proposition 3. *There is no* $\mathcal{L}_{\exp}$*-embedding of* $\mathcal{H}_{\exp}$ *into* L.

Proof. Assume for contradiction that there is such an embedding Ψ and write $\xi := \Psi(\mathrm{id})$. The field L is contained in $\mathcal{H}_{\exp}$, and since Ψ must commute with semialgebraic functions as well as with exp and log, we have $\Psi(L) = L \circ \xi$. For the same reasons, we have $\Psi(L \circ \xi^{\mathrm{inv}}) = L$. The function Ψ is injective, so $L \circ \xi^{\mathrm{inv}}$ must coincide with $\mathcal{H}_{\exp}$, whence in $L = \mathcal{H}_{\exp} \circ \xi = \mathcal{H}_{\exp}$. But this is known to be false: for instance it is a theorem of van den Dries, Macintyre and Marker that the germ $f \in \mathcal{H}_{\exp}$ of the functional inverse of $\log \cdot (\log \circ \log)$ does not lie in L. □

This raises a question:

Question 3. Is there a Liouville-closed Hardy field which is a model of $\mathbb{R}_{\exp}$? Is there an H-closed Hardy field which is a model of $\mathbb{R}_{\exp}$?

3 Second question

The answer to the second question is positive. In order to $\mathcal{L}_{\exp}$-embed $\mathcal{H}$ into an elementary extension of $\mathbb{R}_{\exp}$, it is enough, since $\mathcal{H} \supseteq \mathbb{R}$ and by model completeness, to construe it as a substructure of a model of $\mathrm{Th}(\mathbb{R}_{\exp})$.

Proposition 4. *Let $\mathcal{H}$ be a Hardy field containing* $\mathbb{R}$ *and closed under* exp. *Then* $(\mathcal{H},+,\cdot,\exp)$ *embeds into an elementary extension of* $\mathbb{R}_{\exp}$.

Proof. We need to prove that $\mathcal{H}$ is a model of the universal theory $T_{\forall,\exp}$ of $T_{\exp}$. Consider a universal formula ψ: $\forall\overline{v}(\varphi[\overline{v}])$ where $\varphi[\overline{v}]$ is a boolean combination of atomic $\mathcal{L}_{\exp}$-formulas, hence, up to equivalence modulo the theory of rings, of exponential-polynomial equations, inequations, inequalities… We can assume that $\varphi[\overline{v}]$ is in disjunctive conjunctive form

$$\varphi[\overline{v}]\colon \bigvee_{i\in I}\bigwedge_{j\in J} P_{i,j}(v_1,\ldots,v_n,\mathrm{e}^{v_1},\ldots,\mathrm{e}^{v_n})\,\square_{i,j}\,0,$$

where each $P_{i,j}\in\mathbb{R}[X_1,\ldots,X_{2n}]$, each symbol $\square_{i,j}$ is among $=,<,$ and $>$, and I,J are finite sets.

Assume that ψ is valid in $\mathbb{R}_{\exp}$ and let $f_1,\ldots,f_n$ be representatives of germs in $\mathcal{H}$. Since I is finite, there are a cofinal subset $X\subseteq\mathbb{R}$ and an $i\in I$ for which we have

$$\bigwedge_j P_{i,j}(f_1(t),\ldots,f_n(t),\mathrm{e}^{f_1(t)},\ldots,\mathrm{e}^{f_n(t)})\,\square_{i,j}\,0$$

whenever $t\in X$. Since $\mathcal{H}$ is a Hardy field, the sign of each function

$$t\mapsto P_{i,j}(f_1(t),\ldots,f_n(t),\mathrm{e}^{f_1(t)},\ldots,\mathrm{e}^{f_n(t)})$$

for $j\in J$ is stationnary. So we actually have $\bigwedge_j P_{i,j}(f_1(t),\ldots,f_n(t),\mathrm{e}^{f_1(t)},\ldots,\mathrm{e}^{f_n(t)})\,\square_{i,j}\,0$ for all sufficiently large $t\in\mathbb{R}$. This means that $\mathcal{H}\vDash\bigwedge_j P_{i,j}(f_1,\ldots,f_n,\mathrm{e}^{f_1},\ldots,\mathrm{e}^{f_n})\,\square_{i,j}\,0$, whence in particular that $\mathcal{H}\vDash\varphi[f_1,\ldots,f_n]$. Therefore ψ is valid in $\mathcal{H}$. This shows that $\mathcal{H}\vDash T_{\forall,\exp}$ embeds into a model of $T_{\exp}$. □

Corollary 3. *Every Hardy field $\mathcal{H}$ closed under* exp *embeds into* $(\mathbf{No},\exp)$ *as an ordered exponential field.*

Proof. We first embed $\mathcal{H}$ into a Hardy field $\mathcal{H}^*\supseteq\mathbb{R}$ which is closed under exp. This is just done by closing $\mathcal{H}(\mathbb{R})$ under exp as explained in Lou's lecture. Then embed $\mathcal{H}^*$ into a model of $T_{\exp}$ using Proposition 4, and conclude with Corollary 2. □

Bibliography

[1] H. Gonshor. *An Introduction to the Theory of Surreal Numbers*. Cambridge Univ. Press, 1986.

[2] L. van den Dries and P. Ehrlich. Fields of surreal numbers and exponentiation. *Fundamenta Mathematicae*, 167(2):173–188, 2001.